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Stability of scaling regimes in $d \geqslant 2$ developed turbulence with weak anisotropy

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Fully developed turbulence with weak anisotropy is investigated by means of the renormalization group approach and double expansion regularization for dimensions $d \geqslant 2$. Some modification of the standard minimal subtraction scheme has been used to analyze the stability of the Kolmogorov scaling regime which is governed by the renormalization group fixed point. This fixed point is unstable at $d=2$; thus the infinitesimally weak anisotropy destroys the above scaling regime in two-dimensional space. The restoration of the stability of this fixed point, under a transition from $d=2$ to 3, is demonstrated at a borderline dimension $2<d_c<3$. The results are in qualitative agreement with results recently obtained in the framework of a typical analytical regularization scheme.

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I. INTRODUCTION

A traditional approach to the description of fully developed turbulence is based on the stochastic Navier-Stokes equation [1]. The complexity of this equation leads to great difficulties, which do not allow one to solve it even in the simplest case when one assumes the isotropy of the system under consideration. On the other hand, isotropic turbulence is almost a delusion, and if it exists it is still rather rare. Therefore, if one wants to model more or less realistically developed turbulence, one is pushed to consider anisotropically forced turbulence rather than isotropically forced turbulence. This, of course, rapidly increases the complexity of the corresponding differential equation, which itself has to involve the part responsible for a description of the anisotropy. An exact solution of the stochastic Navier-Stokes equation does not exist, and one is forced to find some convenient methods to treat the problem at least step by step.

A suitable and also powerful tool in the theory of developed turbulence is the so-called renormalization group (RG) method.[1] During the last two decades the RG technique was widely used in this field of science, and gives answers to some principal questions (e.g., the fundamental description of the infrared (IR) scale invariance), and is also useful for calculations of many universal parameters (e.g., critical dimensions of the fields and their gradiens, etc.). A detailed survey of these questions can be found in Refs. [4–6], and references therein.

In early papers, the RG approach was applied only to isotropic models of developed turbulence. However, the method can also be used (with some modifications) in the theory of anisotropically developed turbulence. A crucial question immediately arises here: whether the principal properties of the isotropic case and the anisotropic case are the same, at least at the qualitative level. If they are, then it is possible to consider the isotropic case as a first step in the investigation of real systems. In this type of transition from isotropically developed turbulence into the anisotropically developed turbulence, we have to learn whether the scaling regime remains stable under this transition. In other words, do the stable fixed points of the RG equations remain stable under the influence of anisotropy?

During the last decade a few papers have appeared in which the above question was considered. In some cases, it was found that stability actually takes place (see, e.g., Refs. [7,8]). On the other hand, the existence of systems without such a stability has also been proven. As shown in Ref. [9], in anisotropically magnetohydrodynamically developed turbulence a stable regime generally does not exist. In Refs. [8,10], d-dimensional models with $d>2$ were investigated for two cases, weak anisotropy [8] and strong anisotropy [10], and it was shown that the stability of the isotropic fixed point is lost for dimensions $d<d_c=2.68$. It was also shown that the stability of the fixed point, even for dimension $d=3$, takes place only for sufficiently weak anisotropy. The only problem in these investigations is that it is impossible to use them in the case $d=2$, because new ultraviolet (UV) divergences appear in the Green functions, when one considers $d=2$, and they were not taken into account in Refs. [8,10].

In Ref. [11], a correct treatment of the two-dimensional isotropic turbulence was given. The correctness in the renormalization procedure was reached by introducing a new local term (with a new coupling constant) into the models, which allows one to remove additional UV divergences. From this point of view, the results obtained earlier for anisotropically developed turbulence, presented in Ref. [12] and based on Ref. [13] (the results of the last paper are in conflict with Ref. [11]) cannot be considered as correct because they are inconsistent with the basic requirement of the UV renormalization, namely, with the requirement of the localness of the counterterms [14,15].

The authors of a recent paper [16] used the double-expansion procedure introduced in Ref. [11] (this procedure is a combination of the well-known Wilson dimensional regularization procedure and an analytical one) and the minimal substraction (MS) scheme [17] for an investigation of developed turbulence with weak anisotropy for $d=2$. In such

[1]Here we consider the quantum-field renormalization group approach [2] instead of the Wilson renormalization group technique [3].

a perturbative approach the deviation of the spatial dimension from $d=2$ and $\delta=(d-2)/2$, and that of the exponent of the powerlike correlation function of random forcing from their critical values ϵ, play the roles of expansion parameters.

The main result of the paper was the conclusion that the two-dimensional (2D) fixed point is not stable under weak anisotropy. It means that 2D turbulence is very sensitive to the anisotropy, and no stable scaling regimes exist in this case. In the case $d=3$, for both isotropic and anisotropic turbulence, as mentioned above, the existence of a stable fixed point, which governs the Kolmogorov asymptotic regime, was established by means of the RG approach by using the analytical regularization procedure [7,8,10]. One can perform an analytical continuation from $d=2$ to the three-dimensional turbulence (in the same sense as in the theory of critical phenomena), and verify whether the stability of the fixed point (or, equivalently, the stability of the Kolmogorov scaling regime) is restored. From the analysis made in Ref. [16], it follows that it is impossible to restore the stable regime by transition from dimension $d=2$ to 3. We suppose that the main reason for the above described discrepancy is related to the straightforward application of the standard MS scheme. In the standard MS scheme one works with a purely divergent part of the Green functions only, and in concrete calculations its dependence on the space dimension d, resulting from the tensor nature of these Green functions, is neglected (see Sec. III). In the case of isotropic models, the stability of the fixed points is independent of dimension d. However, in anisotropic models the stability of fixed points depends on the dimension d, and consideration of the tensor structure of Feynman graphs in the analysis of their divergences becomes important.

In the present paper we suggest applying a modified MS scheme in which we keep the d dependence of the UV divergences of graphs. We affirm that after such a modification the d dependence is correctly taken into account, and can be used to investigate whether it is possible to restore the stability of the anisotropically developed turbulence for some dimension d_c when going from a two-dimensional system to a three-dimensional one. In the limit of infinitesimally weak anisotropy for the physically most reasonable value of $\epsilon=2$, the value of the borderline dimension is $d_c=2.44$. Below the borderline dimension, the stable regime of the fixed point of the isotropically developed turbulence is lost by influence of weak anisotropy.

It should be mentioned that a similar idea of a ''geometric factor'' was used in Ref. [18] in a RG analysis of the Burgers-Kardar-Parisi-Zhang equation, but the reason for keeping the d dependence of divergent parts of the graphs was to take correctly into account the finite part of the one-loop Feynman diagrams in the two-loop approximation. In the present paper, we shall not discuss this in detail, because the critical analysis of the results obtained in Ref. [18] was given in Ref. [19].

The paper is organized as follows. In Sec. II we give the quantum field functional formulation of the problem of fully developed turbulence with weak anisotropy. A RG analysis is given in Sec. III, where we discuss the stability of the fixed point obtained under weak anisotropy. In Sec. IV we discuss our results. Appendix A contains expressions for the divergent parts of the important graphs. Finally, Appendix B contains analytical expressions for the fixed point, and the equation which describes its stability in the limit of weak anisotropy.

II. DESCRIPTION OF THE MODEL: UV DIVERGENCIES

In this section we give a description of the model. As already discussed in Sec. I, we work with fully developed turbulence, and assume a weak anisotropy of the system. This means that the parameters that describe deviations from the fully isotropic case are sufficiently small, and allow one to forget about corrections of higher degrees (than linear) which are made by them.

In the statistical theory of anisotropically developed turbulence, the turbulent flow can be described by a random velocity field $\vec{v}(\vec{x},t)$, and its evolution is given by the randomly forced Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}-\nu_0\Delta\vec{v}-\vec{f}^A=\vec{f}, \tag{2.1}$$

where we assume the incompressibility of the fluid, which is given mathematically by the well-known conditions $\vec{\nabla}\cdot\vec{v}=0$ and $\vec{\nabla}\cdot\vec{f}=0$. In Eq. (2.1) the parameter ν_0 is the kinematic viscosity (hereafter all parameters with a subscript 0 denote bare parameters of unrenormalized theory; see below); the term $\vec{f}^A$ is related to anisotropy, and will be specified later. The large-scale random force per unit mass $\vec{f}$ is assumed to have Gaussian statistics defined by the averages

$$\langle f_i\rangle=0, \quad \langle f_i(\vec{x}_1,t)f_j(\vec{x}_2,t)\rangle=D_{ij}(\vec{x}_1-\vec{x}_2,t_1-t_2). \tag{2.2}$$

The two-point correlation matrix

$$D_{ij}(\vec{x},t)=\delta(t)\int\frac{d^d\vec{k}}{(2\pi)^d}\tilde{D}_{ij}(\vec{k})\exp(i\vec{k}\cdot\vec{x}) \tag{2.3}$$

is convenient to parametrize as [7,9]

$$\tilde{D}_{ij}(\vec{k})=g_0\nu_0^3k^{4-d-2\epsilon}[(1+\alpha_{10}\xi_k^2)P_{ij}(\vec{k})+\alpha_{20}R_{ij}(\vec{k})], \tag{2.4}$$

where a vector $\vec{k}$ is the wave vector, d is the dimension of the space (in our case, $2\leqslant d$), and $\epsilon\geqslant 0$ is a dimensionless parameter of the model. If the dimension of the system is taken as $d>2$, then the physical value of this parameter is $\epsilon=2$ (the so-called energy pumping regime). The situation is more complicated when $d=2$. In this case new integrals of motion arise, namely, the enstrophy, and all its powers (for details, see Ref. [20]) which leads to ambiguity in the determination of the inertial range, and this freedom is given in the RG method by the value of the parameter ϵ. The value $\epsilon=3$ corresponds to the so-called enstrophy pumping regime. This problem of uncertainty cannot be solved in the framework of the RG technique. On the other hand, the value of ϵ is not important for the stability of the fixed point when $d=2$. Thus, from our point of view, the value of ϵ in the case $d=2$ is not important. Its value $\epsilon=0$ corresponds to a logarithmic perturbation theory for a calculation of the Green functions when g_0, which plays the role of a bare coupling constant of the model, becomes dimensionless. The problem of continuation from $\epsilon=0$ to physical values was discussed in Ref. [21]. The $(d\times d)$ matrices P_{ij} and R_{ij} are the transverse projection operators, and in wave-number space are defined by the relations

$$P_{ij}(\vec{k})=\delta_{ij}-\frac{k_i k_j}{k^2},\quad R_{ij}(\vec{k})=\left(n_i-\xi_k\frac{k_i}{k}\right)\left(n_j-\xi_k\frac{k_j}{k}\right), \tag{2.5}$$

where ξ_k is given by the equation $\xi_k=\vec{k}\cdot\vec{n}/k$. In Eq. (2.5), the unit vector $\vec{n}$ specifies the direction of the anisotropy axis. The tensor $\tilde{D}_{ij}$, given by Eq. (2.4), is the most general form with respect to the condition of incompressibility of the system under consideration, and contains two dimensionless free parameters α_{10} and α_{20}. From the positiveness of the correlator tensor D_{ij}, one immediately obtains restrictions on the above parameters, namely, $\alpha_{10}\geqslant -1$ and $\alpha_{20}\geqslant -1$. In what follows we assume that these parameters are small enough, and generate only small deviations from the isotropy case.

Using the well-known Martin-Siggia-Rose formalism of the stochastic quantization [22–25], one can transform the stochastic problem [Eq. (2.1)] with the correlator [Eq. (2.3)] into the quantum field model of fields $\vec{v}$ and $\vec{v}\,'$. Here $\vec{v}\,'$ is independent of the $\vec{v}$ auxiliary incompressible field, which we have to introduce when transforming the stochastic problem into a functional form.

The action of the fields $\vec{v}$ and $\vec{v}\,'$ is given in the form

$$\begin{aligned}S=&\frac{1}{2}\int d^d\vec{x}_1\,dt_1\,d^d\vec{x}_2\,dt_2[v_i'(\vec{x}_1,t_1)\\&\times D_{ij}(\vec{x}_1-\vec{x}_2,t_1-t_2)v_j'(\vec{x}_2,t_2)]+\int d^d\vec{x}\,dt\{\vec{v}\,'(\vec{x},t)\\&\times[-\partial_t\vec{v}-(\vec{v}\cdot\vec{\nabla})\vec{v}+\nu_0\vec{\nabla}^2\vec{v}+\vec{f}^A](\vec{x},t)\}.\end{aligned} \tag{2.6}$$

The functional formulation gives the possibility of using quantum field theory methods, including the RG technique to solve the problem. By means of the RG approach it is possible to extract large-scale asymptotic behavior of the correlation functions after an appropriate renormalization procedure which is needed to remove UV divergences.

Now we can return to give an explicit form of the anisotropic dissipative term $\vec{f}^A$. When $d>2$, the UV divergences are only present in the one-particle-irreducible Green function $\langle\vec{v}\,'\vec{v}\rangle$. To remove them, one needs to introduce into the action, in addition to the counterterm $\vec{v}\,'\vec{\nabla}^2\vec{v}$ (the only counterterm needed in the isotropic model), the terms $\vec{v}\,'(\vec{n}\cdot\nabla)^2\vec{v}$, $(\vec{n}\cdot\vec{v}\,')\vec{\nabla}^2(\vec{n}\cdot\vec{v})$, and $(\vec{n}\cdot\vec{v}\,')(\vec{n}\cdot\vec{\nabla})^2(\vec{n}\cdot\vec{v})$. These additional terms are needed to remove divergences related to anisotropic structures. In this case $(d>2)$, one can use the above action [Eq. (2.6)] with [Eq. (2.4)] to solve the anisotropic turbulent problem. Therefore, in order to arrive at a multiplicatively renormalizable model, we have to take the term $\vec{f}^A$ in the form

$$\vec{f}^A=\nu_0[\chi_{10}(\vec{n}\cdot\vec{\nabla})^2\vec{v}+\chi_{20}\vec{n}\vec{\nabla}^2(\vec{n}\cdot\vec{v})+\chi_{30}\vec{n}(\vec{n}\cdot\vec{\nabla})^2(\vec{n}\cdot\vec{v})]. \tag{2.7}$$

Bare parameters χ_{10}, χ_{20}, and χ_{30} characterize the weight of the individual structures in Eq. (2.7).

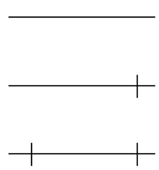
A more complicated situation arises in the specific case $d=2$, where new divergences appear. They are related to the 1-irreducible Green function $\langle\vec{v}\,'\vec{v}\,'\rangle$, which is finite when $d>2$. Here one comes to the problem of how to remove these divergences, because the term in our action, which contains a structure of this type, is nonlocal, namely, $\vec{v}\,'k^{4-d-2\epsilon}\vec{v}\,'$. The only correct way of solving the above problem is to introduce a new local term of the form $\vec{v}\,'\vec{\nabla}^2\vec{v}\,'$ (isotropic case) into the action [11]. In the anisotropic case, we have to introduce additional counterterms $\vec{v}\,'(\vec{n}\cdot\nabla)^2\vec{v}\,'$, $(\vec{n}\cdot\vec{v}\,')\vec{\nabla}^2(\vec{n}\cdot\vec{v}\,')$ and $(\vec{n}\cdot\vec{v}\,')(\vec{n}\cdot\vec{\nabla})^2(\vec{n}\cdot\vec{v}\,')$. In Refs. [11,13] a double-expansion method with a simultaneous deviation $2\delta=d-2$ from the spatial dimension $d=2$, and also a deviation ϵ from the k^2 form of the forcing pair correlation function proportional to $k^{2-2\delta-2\epsilon}$, were proposed. We shall follow the formulation founded on the two-expansion parameters in the present paper.

In this case, the kernel [Eq. (2.4)] corresponding to the correlation matrix $D_{ij}(\vec{x}_1-\vec{x}_2,t_2-t_1)$ in action (2.6) is replaced by the expression

$$\begin{aligned}\tilde{D}_{ij}(\vec{k})=&g_{10}\nu_0^3k^{2-2\delta-2\epsilon}[(1+\alpha_{10}\xi_k^2)P_{ij}(\vec{k})+\alpha_{20}R_{ij}(\vec{k})]\\&+g_{20}\nu_0^3k^2[(1+\alpha_{30}\xi_k^2)P_{ij}(\vec{k})\\&+(\alpha_{40}+\alpha_{50}\xi_k^2)R_{ij}(\vec{k})].\end{aligned} \tag{2.8}$$

Here P_{ij} and R_{ij} are given by relations (2.5); g_{20}, α_{30}, α_{40}, and α_{50} are new parameters of the model; and the parameter g_0 in Eq. (2.4) is now renamed g_{10}. One can see that in such a formulation the counterterm $\vec{v}\,'\vec{\nabla}^2\vec{v}\,'$ and all anisotropic terms can be taken into account by a renormalization of the coupling constant g_{20}, and the parameters α_{30}, α_{40}, and α_{50}.

It has to be stressed that the last term of the $\vec{f}^A$ in Eq. (2.7), which is characterized by the parameter χ_{30}, and the term of the correlation matrix [Eq. (2.8)], related to the parameter α_{50}, are of the order $O(n^4)$, in contrast to the others which are either $O(n^0)$ (the isotropic terms) or $O(n^2)$. Because we work in the limit of weak anisotropy, this fact causes, as a consequence, the values at the fixed point to vanish. On the other hand, the eigenvalues of the stability matrix, which correspond to the parameters χ_{30} and α_{50}, are of the same order, $O(\varepsilon)$, as the eigenvalues which correspond to the other parameters; they play important roles in the determination of stability of the regime (see details in Sec. III).



$$= \langle v_i v_j \rangle_0 \equiv \Delta_{ij}^{vv}(\vec{k},\omega_k)$$

$$= \langle v_i v'_j \rangle_0 \equiv \Delta_{ij}^{vv'}(\vec{k},\omega_k)$$

$$= \langle v'_i v'_j \rangle_0 \equiv \Delta_{ij}^{v'v'}(\vec{k},\omega_k) = 0$$

FIG. 1. The propagators of the model.

Action (2.6), with the kernel $\tilde{D}_{ij}(\vec{k})$ [Eq. (2.8)], is given in a form convenient for a realization of the quantum field perturbation analysis with the standard Feynman diagram technique. From the quadratic part of the action, one obtains the matrix of bare propagators. Their wave-number–frequency representation is presented in Fig. 1, where

$$\Delta_{ij}^{vv}(\vec{k},\omega_k) = -\frac{K_3}{K_1K_2}P_{ij} + \frac{1}{K_1[K_2+\tilde{K}(1-\xi_k^2)]} \times\left[\frac{\tilde{K}K_3}{K_2} + \frac{\tilde{K}[K_3+K_4(1-\xi_k^2)]}{[K_1+\tilde{K}(1-\xi_k^2)]} - K_4\right]R_{ij},$$

$$\Delta_{ij}^{vv'}(\vec{k},\omega_k) = \frac{1}{K_2}P_{ij} - \frac{\tilde{K}}{K_2[K_2+\tilde{K}(1-\xi_k^2)]}R_{ij}, \quad (2.9)$$

with

$$K_1 = i\omega_k + \nu_0 k^2 + \nu_0\chi_{10}(\vec{n}\cdot\vec{k})^2,$$

$$K_2 = -i\omega_k + \nu_0 k^2 + \nu_0\chi_{10}(\vec{n}\cdot\vec{k})^2,$$

$$K_3 = -g_{10}\nu_0^3 k^{2-2\delta-2\epsilon}(1+\alpha_{10}\xi_k^2) - g_{20}\nu_0^3 k^2(1+\alpha_{30}\xi_k^2),$$

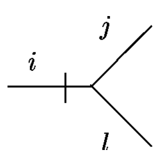
$$K_4 = -g_{10}\nu_0^3 k^{2-2\delta-2\epsilon}\alpha_{20} - g_{20}\nu_0^3 k^2(\alpha_{40}+\alpha_{50}\xi_k^2),$$

$$\tilde{K} = \nu_0\chi_{20}k^2 + \nu_0\chi_{30}(\vec{n}\cdot\vec{k})^2. \quad (2.10)$$

The propagators are written in a form suitable for strong anisotropy when the parameters α_{i0} are not small. In the case of weak anisotropy, it is possible to make an expansion, and to work only with linear terms with respect to all parameters which characterize anisotropy. The interaction vertex in our model is given in Fig. 2. Here the wave vector $\vec{k}$ corresponds to the field $\vec{v}'$. Now one can use the above introduced Feynman rules for a computation of all needed graphs.

III. RG ANALYSIS AND STABILITY OF THE FIXED POINT

Using the standard analysis of quantum field theory (see, e.g., Refs. [4,6,14,15]), one can find that the UV divergences



$$\equiv V_{ijl} = i(k_j\delta_{il} + k_l\delta_{ij}).$$

FIG. 2. The vertex of the model.

of one-particle-irreducible Green functions $\langle vv'\rangle_{IR}$ and $\langle v'v'\rangle_{IR}$ are quadratic in the wave vector. The last one takes place only in the case when the dimension of the space is 2. All terms needed for removing the divergences are included in action (2.6), with Eq. (2.7) and kernel (2.8). This leads to the fact that our model is multiplicatively renormalizable. Thus one can immediately write down the renormalized action in the wave-number–frequency representation, with $\vec{\nabla} \to i\vec{k}, \partial_t \to -i\omega_k$ (all needed integrations and summations are assumed):

$$S^R(v,v') = \tfrac{1}{2}v'_i\{g_1\nu^3\mu^{2\epsilon}k^{2-2\delta-2\epsilon}[(1+\alpha_1\xi_k^2)P_{ij}+\alpha_2R_{ij}] + g_2\nu^3\mu^{-2\delta}k^2[(Z_5+Z_6\alpha_3\xi_k^2)P_{ij} + (Z_7\alpha_4+Z_8\alpha_5\xi_k^2)R_{ij}]\}v'_j + v'_i\{(i\omega_k - Z_1\nu k^2)P_{ij} - \nu k^2[Z_2\chi_1\xi_k^2P_{ij} + (Z_3\chi_2+Z_4\chi_3\xi_k^2)R_{ij}]\}v_j + \tfrac{1}{2}v'_iv_jv_lV_{ijl}, \quad (3.1)$$

where μ is a scale setting parameter with the same canonical dimension as the wave number. Quantities g_i, χ_i, α_3, α_4, α_5, and ν are the renormalized counterparts of bare ones, and Z_i are renormalization constants which are expressed via the UV divergent parts of the functions $\langle vv'\rangle_{IR}$ and $\langle v'v'\rangle_{IR}$. Their general form in the one loop approximation is

$$Z_i = 1 - F_i\mathcal{P}_i^{\delta,\epsilon}. \quad (3.2)$$

In the standard MS scheme the amplitudes F_i are only some functions of g_i, χ_i, α_3, α_4, α_5, and are independent of d and ϵ. The terms $\mathcal{P}_i^{\delta,\epsilon}$ are given by linear combinations of the poles $1/2\epsilon$, $1/2\delta$, and $1/(4\epsilon+2\delta)$ (for $\delta\to 0$ and $\epsilon\to 0$). The amplitudes $F_i = F_i^{(1)}F_i^{(2)}$ are a product of two multipliers $F_i^{(1)}$ and $F_i^{(2)}$. One of them, say, $F_i^{(1)}$ is a multiplier originating from the divergent part of the Feynman diagrams, and the second one, $F_i^{(2)}$, is connected only with the tensor nature of the diagrams. We explain this using the following simple example. Consider a UV-divergent integral

$$I(\mathbf{k},\mathbf{n}) \equiv n_in_jk_lk_m\int d^d\mathbf{q}\frac{1}{(q^2+m^2)^{1+2\delta}} \times\left(\frac{q_iq_jq_lq_m}{q^4} - \frac{\delta_{ij}q_lq_m+\delta_{il}q_jq_m+\delta_{jl}q_iq_m}{3q^2}\right)$$

(summations over repeated indices are implied), where m is an infrared mass. It can be simplified as

$$I(\mathbf{k},\mathbf{n}) \equiv n_in_jk_lk_mS_{ijlm}\int_0^\infty dq^2\frac{q^{2\delta}}{2(q^2+m^2)^{1+2\delta}},$$

where

$$S_{ijlm}=\frac{S_d}{d(d+2)}\left(\delta_{ij}\delta_{lm}+\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}\right.$$
$$\left.-\frac{(d+2)}{3}(\delta_{ij}\delta_{lm}+\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})\right),$$

$$\int_0^\infty dq^2\frac{q^{2\delta}}{2(q^2+m^2)^{1+2\delta}}=\frac{\Gamma(\delta+1)\Gamma(\delta)}{2m^{2\delta}\Gamma(2\delta+1)},$$

and $S_d=2\pi^{d/2}/\Gamma(d/2)$ is the surface of unit the d-dimensional sphere. The purely UV divergent part manifests itself as the pole in $2\delta=d-2$; therefore, we find

$$\text{UVdiv. part of } I=\frac{1}{2\delta}[F_1^{(2)}k^2+F_2^{(2)}(\mathbf{nk})^2],$$

where $F_1^{(2)}=F_2^{(2)}/2=(1-d)S_d/3d(d+2)$ $(F_1^{(1)}=F_2^{(1)}=1)$.

In the standard MS scheme one puts $d=2$ in $F_1^{(2)}$ and $F_2^{(2)}$; therefore, the d dependence of these multipliers is ignored. For theories with vector fields and, consequently, with tensor diagrams, where the sign of values of fixed points and/or their stability depend on the dimension d, the procedure, which eliminates the dependence of multipliers of the types $F_1^{(2)}$ and $F_2^{(2)}$ on d, is not completely correct because one is not able to control the stability of the fixed point when $d=3$. In the analysis of Feynman diagrams, we propose to slightly modify the MS scheme in such a way that we keep the d dependence of F in Eq. (3.2). The following calculations of RG functions (β functions and anomalous dimensions) allow one to arrive at results which are in qualitative agreement with the results obtained recently in the framework of the simple analytical regularization scheme [10], i.e. we obtain a fixed point which is not stable for $d=2$, but whose stability is restored for a borderline dimension $2<d_c<3$.

The transition from action (2.6) to the renormalized one [Eq. (3.1)] is given by the introduction of the renormalization constants Z,

$$\nu_0=\nu Z_\nu,\quad g_{10}=g_1\mu^{2\epsilon}Z_{g_1},\quad g_{20}=g_2\mu^{-2\delta}Z_{g_2},$$
$$\chi_{i0}=\chi_i Z_{\chi_i},\quad \alpha_{(i+2)0}=\alpha_{i+2}Z_{\alpha_{i+2}},\tag{3.3}$$

where $i=1$, 2, and 3. By comparison of the corresponding terms in action (3.1) with definitions of the renormalization constants Z for parameters (3.3), one can immediately write down relations between them. We have

$$Z_\nu=Z_1,$$
$$Z_{g_1}=Z_1^{-3},$$
$$Z_{g_2}=Z_5Z_1^{-3},\tag{3.4}$$
$$Z_{\chi_i}=Z_{1+i}Z_1^{-1},$$
$$Z_{\alpha_{i+2}}=Z_{i+5}Z_5^{-1},$$

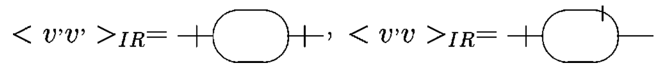


FIG. 3. Divergent one-irreducible Green functions in the one-loop approximation.

where again $i=1$, 2, and 3.

In the one-loop approximation, divergent one-irreducible Green functions $\langle v'v\rangle_{IR}$ and $\langle v'v'\rangle_{IR}$ are represented by the Feynman graphs, which are shown in Fig. 3. The divergent parts of these diagrams $\Gamma^{v'v'}$ and $\Gamma^{v'v}$ have the structure

$$\Gamma^{v'v'}=\frac{1}{2}\nu^3A\left[\frac{g_1^2}{4\epsilon+2\delta}[a_1\delta_{ij}k^2+a_2\delta_{ij}(\vec{n}\cdot\vec{k})^2+a_3n_in_jk^2\right.$$
$$+a_4n_in_j(\vec{n}\cdot\vec{k})^2]+\frac{g_1g_2}{2\epsilon}[b_1\delta_{ij}k^2+b_2\delta_{ij}(\vec{n}\cdot\vec{k})^2$$
$$+b_3n_in_jk^2+b_4n_in_j(\vec{n}\cdot\vec{k})^2]+\frac{g_2^2}{-2\delta}[c_1\delta_{ij}k^2$$
$$\left.+c_2\delta_{ij}(\vec{n}\cdot\vec{k})^2+c_3n_in_jk^2+c_4n_in_j(\vec{n}\cdot\vec{k})^2]\right],$$

$$\Gamma^{v'v}=-\nu A\left[\frac{g_1}{2\epsilon}[d_1\delta_{ij}k^2+d_2\delta_{ij}(\vec{n}\cdot\vec{k})^2+d_3n_in_jk^2\right.$$
$$+d_4n_in_j(\vec{n}\cdot\vec{k})^2]+\frac{g_2}{-2\delta}[e_1\delta_{ij}k^2+e_2\delta_{ij}(\vec{n}\cdot\vec{k})^2$$
$$\left.+e_3n_in_jk^2+e_4n_in_j(\vec{n}\cdot\vec{k})^2]\right],\tag{3.5}$$

where parameter A and functions a_i, b_i, c_i, d_i, and e_i are given in Appendix A $(i=1,\ldots,4)$. The counterterms are built up from these divergent parts, which lead to the following equations for renormalization constants:

$$Z_1=1-A\left(\frac{g_1}{2\epsilon}d_1-\frac{g_2}{2\delta}e_1\right),$$
$$Z_{1+i}=1-\frac{A}{\chi_i}\left(\frac{g_1}{2\epsilon}d_{1+i}-\frac{g_2}{2\delta}e_{1+i}\right),$$
$$Z_5=1+\frac{A}{2}\left(\frac{g_1^2}{g_2}\frac{a_1}{4\epsilon+2\delta}+\frac{g_1}{2\epsilon}b_1-\frac{g_2}{2\delta}c_1\right),\tag{3.6}$$
$$Z_{5+i}=1+\frac{A}{2\alpha_{i+2}}\left(\frac{g_1^2}{g_2}\frac{a_{i+1}}{4\epsilon+2\delta}+\frac{g_1}{2\epsilon}b_{i+1}-\frac{g_2}{2\delta}c_{i+1}\right),$$
$$i=1,2,3.$$

From these expressions, one can define the corresponding anomalous dimensions $\gamma_i=\mu\partial_\mu\ln Z_i$ for all renormalization constants Z_i (the logarithmic derivative $\mu\partial_\mu$ is taken at fixed values of all bare parameters). The β functions for all invariant charges (running coupling constants g_1 and g_2, and parameters χ_i and α_{i+2}) are given by the relations β_{g_i}

$=\mu\partial_\mu g_i$ ($i=1,2$), $\beta_{\chi_i}=\mu\partial_\mu\chi_i$, and $\beta_{\alpha_{i+2}}=\mu\partial_\mu\alpha_{i+2}$ ($i=1,2,3$). Now using Eqs. (3.4) and definitions given above, one can immediately write the β functions in the forms

$$\begin{aligned}\beta_{g_1}&=-g_1(2\epsilon+\gamma_{g_1})=g_1(-2\epsilon+3\gamma_1),\\ \beta_{g_2}&=g_2(2\delta-\gamma_{g_2})=g_2(2\delta+3\gamma_1-\gamma_5),\\ \beta_{\chi_i}&=-\chi_i\gamma_{\chi_i}=-\chi_i(\gamma_{i+1}-\gamma_1),\\ \beta_{\alpha_{i+2}}&=-\alpha_{i+2}\gamma_{\alpha_{i+2}}=-\alpha_{i+2}(\gamma_{i+5}-\gamma_5),\quad i=1,2,3,\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}\gamma_1&=A(g_1d_1+g_2e_1),\\ \gamma_{i+1}&=\frac{A}{\chi_i}(g_1d_{i+1}+g_2e_{i+1}),\\ \gamma_5&=-\frac{A}{2}\left(\frac{g_1^2}{g_2}a_1+g_1b_1+g_2c_1\right),\\ \gamma_{i+5}&=-\frac{A}{2\alpha_{i+2}}\left(\frac{g_1^2}{g_2}a_{i+1}+g_1b_{i+1}+g_2c_{i+1}\right),\quad i=1,2,3.\end{aligned}\tag{3.8}$$

By substitution of the anomalous dimensions γ_i [Eq. (3.8)] into expressions for the β functions, one obtains

$$\begin{aligned}\beta_{g_1}&=g_1[-2\epsilon+3A(g_1d_1+g_2e_1)],\\ \beta_{g_2}&=g_2\left[2\delta+3A(g_1d_1+g_2e_1)+\frac{A}{2}\left(\frac{g_1^2}{g_2}a_1+g_1b_1+g_2c_1\right)\right],\\ \beta_{\chi_i}&=-A[(g_1d_{i+1}+g_2e_{i+1})-\chi_i(g_1d_1+g_2e_1)],\\ \beta_{\alpha_{i+2}}&=-\frac{A}{2}\left[-\left(\frac{g_1^2}{g_2}a_{i+1}+g_1b_{i+1}+g_2c_{i+1}\right)\right.\\ &\quad\left.+\alpha_{i+2}\left(\frac{g_1^2}{g_2}a_1+g_1b_1+g_2c_1\right)\right],\\ &\quad i=1,2,3.\end{aligned}\tag{3.9}$$

The fixed point of the RG equations is defined by the system of eight equations

$$\beta_C(C_*)=0,\tag{3.10}$$

where we denote $C=\{g_1,g_2,\chi_i,\alpha_{i+2}\}$, $i=1$, 2, and 3, and C_* is the corresponding value for the fixed point. The IR stability of the fixed point is determined by the positive real parts of the eigenvalues of the matrix

$$\omega_{lm}=\left(\frac{\partial\beta_{C_l}}{\partial C_m}\right)_{C=C_*}l,\quad m=1,\ldots,8.\tag{3.11}$$

Now we have all necessary tools at hand to investigate the fixed points and their stability. In *the isotropic case* all parameters which are connected with the anisotropy are equal to zero, and one can immediately find the Kolmogorov fixed point, namely,

$$\begin{aligned}g_{1*}&=\frac{1}{A}\frac{8(2+d)\epsilon[2\epsilon-3d(\delta+\epsilon)+d^2(3\delta+2\epsilon)]}{9(-1+d)^3d(1+d)(\delta+\epsilon)},\\ g_{2*}&=\frac{1}{A}\frac{8(-4-2d+2d^2+d^3)\epsilon^2}{9(-1+d)^3d(1+d)(\delta+\epsilon)},\end{aligned}\tag{3.12}$$

where $\delta=(d-2)/2$ and the corresponding ω_{ij} matrix has the following eigenvalues

$$\begin{aligned}\lambda_{1,2}=&\frac{1}{6d(d-1)}(6d\delta(d-1)+4\epsilon(2-3d+2d^2)\\ &\pm\{[6d\delta(1-d)-4\epsilon(2-3d+2d^2)]^2\\ &-12d(d-1)\epsilon[12d\delta(d-1)+4\epsilon(2-3d+2d^2)]\}^{1/2}).\end{aligned}\tag{3.13}$$

By a detailed analysis of these eigenvalues, we know that in the interesting region of parameters, namely, $\epsilon>0$ and $\delta\geqslant 0$ (it corresponds to $d\geqslant 2$) the above computed fixed point is stable. In the limit $d=2$, this fixed point is in agreement with that given in Refs. [11,16].

When one considers *the weak anisotropy case*, the situation becomes more complicated because of necessity to use all system of β functions if one wants to analyze the stability of the fixed point. It is also possible to find analytical expressions for the fixed point in this more complicated case, because in the weak anisotropy limit it is enough to calculate linear corrections of α_1 and α_2 to all the quantities (see Appendix B).

To investigate the stability of the fixed point, it is necessary to apply it in matrix (3.11). Analysis of this matrix shows us that it can be written in the block-diagonal form: $(6\times6)(2\times2)$. The 2×2 part is given by the β functions of the parameters α_5 and χ_3, and this block is responsible for the existence of the borderline dimension d_c because one of its eigenvalues, say $\lambda_1(\epsilon,d,\alpha_1,\alpha_2)$, has a solution $d_c\in\langle 2,3\rangle$ of the equation $\lambda_1(\epsilon,d_c,\alpha_1,\alpha_2)=0$ for the defined values of ϵ, α_1, and α_2. The following procedure has been used to find the fixed point: First we have used isotropic solution to g_1 and g_2 to compute the expressions for α_{i+2} and χ_i, $i=1$, 2, and 3. From equations $\beta_{\alpha_5}=0$ and $\beta_{\chi_3}=0$, one can immediately find that $\alpha_{5*}=0$ and $\chi_{3*}=0$. After this we can calculate expressions for the fixed point of the parameters α_{i+2} and χ_i, $i=1$ and 2. At the end, we come back to the equations for g_1 and g_2, namely, $\beta_{g_1}=0$ and $\beta_{g_2}=0$, and find linear corrections of α_1 and α_2 to the fixed point. The corresponding expressions for the fixed point and the corresponding eigenvalue of the stability matrix responsible for the instability are given in Appendix B.

For the energy pumping regime ($\epsilon=2$) and $\alpha_1=\alpha_2=0$, we found the critical dimension $d_c=2.44$. This is the case when one supposes only the fact of the anisotropy. Using nonzero values of α_1 and α_2 one can also estimate the influ-

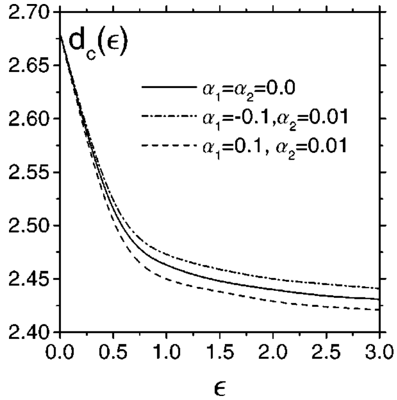


FIG. 4. Dependence of the borderline dimension d_c on the parameter ϵ for concrete values of α_1 and α_2.

ence of these parameters on the borderline dimension d_c. It is interesting to calculate the dependence of d_c on the parameter ϵ too. In Fig. 4, this dependence and the dependence on small values of α_1 and α_2 are presented. As one can see from this figure, d_c increases when $\epsilon \to 0$, and the parameters α_1 and α_2 give small corrections to d_c. In Fig. 5, the dependences of d_c on α_1 and α_2 on $\epsilon=2$ are presented.

IV. CONCLUSION

We have investigated the influence of the weak anisotropy on the fully developed turbulence using the quantum field RG double expansion method, and introduced a modified minimal subtraction scheme in which the space dimension dependence of the divergent parts of the Feynman diagrams is kept. We affirm that such a modified approach is correct when one needs to compute the d dependence of the important quantities, and is necessary for restoration of the stability of scaling regimes when one makes transition from dimension $d=2$ to $d=3$. We have derived analytical expressions for the fixed point in the limit of weak anisotropy, and found an equation which manages the stability of this point as a function of the parameters ϵ, α_1, and α_2, and allows one to calculate the borderline dimension d_c. Below this dimension the fixed point is unstable. In the limit case of infinitesimally small anisotropy ($\alpha_1 \to 0$ and $\alpha_2 \to 0$) and in the energy pumping regime ($\epsilon=2$), we have found the borderline dimension $d_c=2.44$. We have also investigated the ϵ dependence of d_c for different values of the anisotropy parameters α_1 and α_2, and also the dependence of d_c on the relatively small values of α_1 and α_2 for the physical value $\epsilon=2$.

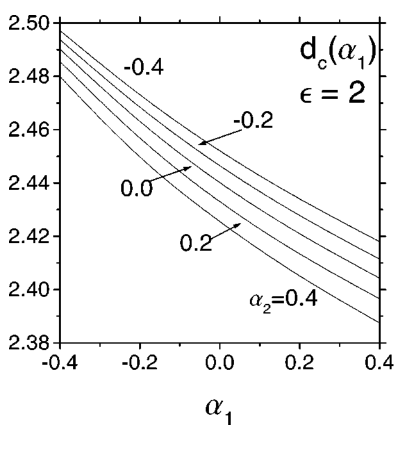


FIG. 5. Dependence of the borderline dimension d_c on the parameters α_1 and α_2 for a physical value $\epsilon=2$.

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APPENDIX A

The explicit form of the parameter A and functions a_i, b_i, c_i, d_i, and e_i ($i=1,\ldots,4$) for the divergent parts of diagrams (Fig. 3)

$$
\begin{aligned}
a_1=&\frac{1}{2d(2+d)(4+d)(6+d)}[-48-20d+70d^2+30d^3\\
&-21d^4-10d^5-d^6+\alpha_2(24+16d-22d^2-16d^3-2d^4)\\
&+\alpha_1(24+52d-4d^2-50d^3-20d^4-2d^5)\\
&+\chi_1(-36-78d+6d^2+75d^3+30d^4+3d^5)\\
&+\chi_2(-36-24d+33d^2+24d^3+3d^4)\\
&+\chi_3(-36-9d+36d^2+9d^3)],
\end{aligned}
$$

$$
\begin{aligned}
a_2=&\frac{1}{4d(2+d)(4+d)(6+d)}[\alpha_1(-96-64d+88d^2+64d^3\\
&+8d^4)+\alpha_2(-96-64d+124d^2+82d^3-26d^4-18d^5\\
&-2d^6)+\chi_1(144+96d-132d^2-96d^3-12d^4)\\
&+\chi_2(144+96d-186d^2-123d^3+39d^4+27d^5+3d^6)\\
&+\chi_3(72+6d-87d^2-9d^3+15d^4+3d^5)],
\end{aligned}
$$

$$a_3=a_2,$$

$$a_4=\frac{6\chi_3(1-d^2)}{(2+d)(6+d)},$$

$$
\begin{aligned}
b_1=&\frac{1}{d(2+d)(4+d)(6+d)}[-48-20d+70d^2+30d^3\\
&-21d^4-10d^5-d^6+\alpha_5(12+3d-12d^2-3d^3)\\
&+(\alpha_2+\alpha_4)(12+8d-11d^2-8d^3-d^4)\\
&+(\alpha_1+\alpha_3)(12+26d-2d^2-25d^3-10d^4-d^5)\\
&+\chi_1(-36-78d+6d^2+75d^3+30d^4+3d^5)\\
&+\chi_2(-36-24d+33d^2+24d^3+3d^4)\\
&+\chi_3(-36-9d+36d^2+9d^3)],
\end{aligned}
$$

$$b_2=\frac{1}{2d(2+d)(4+d)(6+d)}\times[(\alpha_1+\alpha_3)(-48-32d+44d^2+32d^3+4d^4)+\alpha_5(-24-2d+29d^2+3d^3-5d^4-d^5)+(\alpha_2+\alpha_4)(-48-32d+62d^2+41d^3-13d^4-9d^5-d^6)+\chi_1(144+96d-132d^2-96d^3-12d^4)+\chi_2(144+96d-186d^2-123d^3+39d^4+27d^5+3d^6)+\chi_3(72+6d-87d^2-9d^3+15d^4+3d^5)],$$

$$b_3=b_2,$$

$$b_4=\frac{4(d^2-1)(\alpha_5-3\chi_3)}{(2+d)(6+d)},$$

$$c_1=\frac{1}{2d(2+d)(4+d)(6+d)}[-48-20d+70d^2+30d^3-21d^4-10d^5-d^6+\alpha_5(24+6d-24d^2-6d^3)+\alpha_4(24+16d-22d^2-16d^3-2d^4)+\alpha_3(24+52d-4d^2-50d^3-20d^4-2d^5)+\chi_1(-36-78d+6d^2+75d^3+30d^4+3d^5)+\chi_2(-36-24d+33d^2+24d^3+3d^4)+\chi_3(-36-9d+36d^2+9d^3)],$$

$$c_2=\frac{1}{4d(2+d)(4+d)(6+d)}\times[\alpha_3(-96-64d+88d^2+64d^3+8d^4)+\alpha_5(-48-4d+58d^2+6d^3-10d^4-2d^5)+\alpha_4(-96-64d+124d^2+82d^3-26d^4-18d^5-2d^6)+\chi_1(144+96d-132d^2-96d^3-12d^4)+\chi_2(144+96d-186d^2-123d^3+39d^4+27d^5+3d^6)+\chi_3(72+6d-87d^2-9d^3+15d^4+3d^5)],$$

$$c_3=c_2,$$

$$c_4=\frac{(d^2-1)(4\alpha_5-6\chi_3)}{(2+d)(6+d)},$$

$$d_1=\frac{1}{4d(2+d)(4+d)(6+d)}[24d-14d^2-33d^3+13d^4+9d^5+d^6+\alpha_2(12-4d-13d^2+4d^3+d^4)+\alpha_1(-12-20d+3d^2+19d^3+9d^4+d^5)+\chi_1(36+42d-18d^2-40d^3-18d^4-2d^5)+\chi_2(-12+16d+15d^2-16d^3-3d^4)+\chi_3(6+9d-6d^2-9d^3)],$$

$$d_2=\frac{1}{8d(2+d)(4+d)(6+d)}\times[\alpha_1(-48+16d+52d^2-16d^3-4d^4)+\alpha_2(48+80d-60d^2-96d^3+10d^4+16d^5+2d^6)+\chi_1(48-64d-60d^2+64d^3+12d^4)+\chi_2(-48-104d+62d^2+127d^3-11d^4-23d^5-3d^6)+\chi_3(-2d+7d^2+5d^3-7d^4-3d^5)],$$

$$d_3=\frac{1}{8d(2+d)(4+d)(6+d)}\times[\alpha_1(48+56d-40d^2-56d^3-8d^4)+\alpha_2(-48-56d+40d^2+56d^3+8d^4)+\chi_1(-48-104d+32d^2+104d^3+16d^4)+\chi_2(48+32d-38d^2-25d^3-9d^4-7d^5-d^6)+\chi_3(22d-d^2-21d^3+d^4-d^5)],$$

$$d_4=\frac{\chi_3(-10+d+10d^2-d^3)}{2(2+d)(6+d)},$$

$$e_1=\frac{1}{4d(2+d)(4+d)(6+d)}\times[24d-14d^2-33d^3+13d^4+9d^5+d^6+3d\alpha_5(-1+d^2)+\alpha_4(12-4d-13d^2+4d^3+d^4)+\alpha_3(-12-20d+3d^2+19d^3+9d^4+d^5)+\chi_1(36+42d-18d^2-40d^3-18d^4-2d^5)+\chi_2(-12+16d+15d^2-16d^3-3d^4)+\chi_3(6+9d-6d^2-9d^3)],$$

$$e_2=\frac{1}{8d(2+d)(4+d)(6+d)}\times[\alpha_3(-48+16d+52d^2-16d^3-4d^4)+\alpha_5(-8d^2-2d^3+8d^4+2d^5)+\alpha_4(48+80d-60d^2-96d^3+10d^4+16d^5+2d^6)+\chi_1(48-64d-60d^2+64d^3+12d^4)+\chi_2(-48-104d+62d^2+127d^3-11d^4-23d^5-3d^6)+\chi_3(-2d+7d^2+5d^3-7d^4-3d^5)],$$

$$e_3=\frac{1}{8d(2+d)(4+d)(6+d)}\times[24d\alpha_5(-1+d^2)+\alpha_3(48+56d-40d^2-56d^3-8d^4)+\alpha_4(-48-56d+40d^2+56d^3+8d^4)+\chi_1(-48-104d+32d^2+104d^3+16d^4)+\chi_2(48+32d-38d^2-25d^3-9d^4-7d^5-d^6)+\chi_3(22d-d^2-21d^3+d^4-d^5)],$$

$$e_4=\frac{6\alpha_5(1-d^2)+\chi_3(-10+d+10d^2-d^3)}{2(2+d)(6+d)},$$

$$A=\frac{S_d}{(2\pi)^d(d^2-1)},$$

where S_d is a d-dimensional sphere given by the relation

$$S_d=\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}.$$

APPENDIX B

Here we present explicit analytical expressions for the fixed point in the weak anisotropy limit, and also the equation which governs its stability. The basic forms of the fixed point are

$$g_{1*}=g_{10*}+g_{11*}\alpha_1+g_{12*}\alpha_2,$$

$$g_{2*}=g_{20*}+g_{21*}\alpha_1+g_{22*}\alpha_2,$$

$$\alpha_{3*}=e_{11}\alpha_1+e_{12}\alpha_2,$$

$$\alpha_{4*}=e_{21}\alpha_1+e_{22}\alpha_2,$$

$$\chi_{1*}=e_{31}\alpha_1+e_{32}\alpha_2,$$

$$\chi_{2*}=e_{41}\alpha_1+e_{42}\alpha_2,$$

$$\alpha_{5*}=0,$$

$$\chi_{3*}=0,$$

where g_{10*} and g_{20*} are defined in Eq. (3.12), and g_{11*}, g_{12*}, g_{21*}, g_{22*}, and e_{ij}, $i=1$, 2, 3, and 4, $j=1$ and 2, are functions only of the dimension d and parameters ϵ and $\delta=(d-2)/2$. They have the forms

$$g_{11*}=\frac{g_{11n}}{g_{11d}},\quad g_{12*}=\frac{g_{12n}}{g_{12d}},\quad g_{21*}=\frac{g_{21n}}{g_{22d}},\quad g_{22*}=\frac{g_{22n}}{g_{22d}},$$

$$e_{11}=\frac{e_{11n}}{e_d},\quad e_{12}=\frac{e_{12n}}{e_d},\quad e_{21}=\frac{e_{21n}}{e_d},\quad e_{22}=\frac{e_{22n}}{e_d},$$

$$e_{31}=\frac{e_{31n}}{g_se_d},\quad e_{32}=\frac{e_{32n}}{g_se_d},\quad e_{41}=\frac{e_{41n}}{g_se_d},\quad e_{42}=\frac{e_{42n}}{g_se_d},$$

where

$$\begin{aligned}
g_{11n}={}&3(d^2-1)g_{10*}[d^6(g_{10*}+g_{20*})(5e_{31}-3)g_{10*}-3e_{11}g_{20*}+5e_{31}g_{20*}]+3d^5(g_{10*}+g_{20*})[(-2+3e_{31}+2e_{41})g_{10*}\\
&-(2e_{11}+e_{21}-3e_{31}-2e_{41})g_{20*}]-8(g_{10*}+g_{20*})[(-1+3e_{31}-e_{41})g_{10*}-(e_{11}-e_{21}-3e_{31}+e_{41})g_{20*}]\\
&+d^3\{-(g_{10*}+g_{20*})(-4+9e_{31}-6e_{41})g_{10*}+[-4e_{11}+3(e_{21}+3e_{31}-2e_{41})]g_{20*}\}\\
&+8\delta[(-5+10e_{31}+3e_{41})g_{10*}-(5e_{11}+e_{21}-10e_{31}-3e_{41})g_{20*}]\\
&+2d\{-(g_{10*}+g_{20*})[(-1+6e_{41})g_{10*}-(e_{11}+3e_{21}-6e_{41})g_{20}]\}\\
&+16\delta[(-1+3e_{31}-e_{41})g_{10*}-(e_{11}-e_{21}-3e_{31}+e_{41})g_{20*}]+d^2[(g_{10*}+g_{20*})(-15+34e_{31})g_{10*}\\
&+(-15e_{11}+5e_{21}+34e_{31})g_{20*}]+16\delta[(-4+9e_{31}+2e_{41})g_{10*}+(-4e_{11}+9e_{31}+2e_{41})g_{20*}]\\
&+d^4\{8\delta(-g_{10*}+2e_{31}g_{10*}-e_{11}g_{20*}+2e_{31}g_{20*})-(g_{10*}+g_{20*})[(-10+15e_{31}+8e_{41})g_{10*}\\
&+(-10e_{11}-3e21+15e_{31}+8e_{41})g_{20}]\},
\end{aligned}$$

$$\begin{aligned}
g_{11d}={}&2d(4+d)\{-15d^6(g_{10*}+g_{20*})^2+6d^7(g_{10*}+g_{20*})^2+2(g_{10*}+g_{20*})[16\epsilon-3(g_{10*}+g_{20*})]\\
&+4d^4[\epsilon(g_{10*}-2g_{20*})+6(g_{10*}+g_{20*})^2+3\delta(2g_{10*}+g_{20*})]\\
&+d^5[3(g_{10*}+g_{20*})^2+12\delta(2g_{10*}+g_{20*})-4\epsilon(g_{10*}+4g_{20*})]\\
&-4d^3[6(g_{10*}+g_{20*})^2-3\epsilon(g_{10*}+4g_{20*})+\delta[8\epsilon+9(2g_{10*}+g_{20*})]]\}\\
&+d\{15(g_{10*}+g_{20*})^2-8\epsilon(g_{10*}+4g_{20*})+\delta[-128\epsilon+24(2g_{10*}+g_{20*})]\}\\
&-d^2\{4\delta(32\epsilon+6g_{10*}+3g_{20*})+3[(g_{10*}+g_{20*})^2+4\epsilon(3g_{10*}+2g_{20*})]\},
\end{aligned}$$

$$g_{21n}=-[(-1+d^2)(-3)-4-d+4d^2+d^3]g_{10*}[2(-2+d^2)g_{10*}-(4-3d+d^2)g_{20*}]$$
$$\times\{[6e_{31}+d^2(-1+2e_{31})-2(1+e_{41})+3d(-1+2e_{31}+e_{41})]g_{10*}$$
$$-[(2+3d+d^2)e_{11}+(-2+d)e_{21}-6e_{31}-6de_{31}-2d^2e_{31}+2e_{41}-3de_{41})g_{20*}]$$
$$+d(4+d)[-4+6e_{31}+d^2(-2+3e_{31})+6e_{41}+d(-8+12e_{31}+3e_{41})]g_{10*}^2$$
$$+[2+d^2(1-2e_{11})-4e_{11}-4e_{21}-6e_{31}+d(1-8e_{11}-2e_{21}+6e_{31}-3e_{41})+18e_{41}]g_{10*}g_{20*}$$
$$+[(2+d+d^2)e_{11}+(-10+d)e_{21}-3(4e_{31}+2de_{31}+d^2e_{31}-4e_{41}+2de_{41})]g_{20*}^2\}$$
$$\times\{-8(2+d)\epsilon+3[-1+d)^2(1+d)(2g_{10*}+g_{20*})]\},$$

$$g_{21d}=3(-1+d)^2d(4+5d+d^2)(-4-d+4d^2+d^3)g_{10*}[2(-2+d^2)g_{10*}-(4-3d+d^2)g_{20*}]$$
$$+d(4+d)^2[-8(2+d)\epsilon+3(-1+d)^2(1+d)(2g_{10*}+g_{20*})]$$
$$\times[d^2(8\delta+3g_{10*})-4(g_{10*}+g_{20*})-3d^3(g_{10*}+2g_{20*})+d^4(g_{10*}+4g_{20*})+d(16\delta+3g_{10*}+6g_{20*})],$$

$$g_{12n}=3(-1+d^2)g_{10*}(3d^5(g_{10*}+g_{20*})[(-1+3e_{32}+2e_{42})g_{10*}-(2e_{12}+e_{22}-3e_{32}-2e_{42})g_{20*}]$$
$$-8(g_{10*}+g_{20*})[(1+3e_{32}-e_{42})g_{10*}-(e_{12}-e_{22}-3e_{32}+e_{42})g_{20*}]$$
$$+d^6(g_{10*}+g_{20*})[-3e_{12}g_{20*}+5e_{32}(g_{10*}+g_{20*})]+d^3\{-[(g_{10*}+g_{20*})(3(1+3e_{32}-2e_{42})g_{10*}$$
$$+[-4e_{12}+3(e_{22}+3e_{32}-2e_{42})]g_{20*}\})+8\delta[(-1+10e_{32}+3e_{42})g_{10*}-(5e_{12}+e_{22}-10e_{32}-3e_{42})g_{20*}]$$
$$+2d\{-(g_{10*}+g_{20*})[(-3+6e_{42})g_{10*}-(e_{12}+3e_{22}-6e_{42})g_{20*}]\}$$
$$+16\delta[(1+3e_{32}-e_{42})g_{10*}-(e_{12}-e_{22}-3e_{32}+e_{42})g_{20*}]+d^4\{-[(g_{10*}+g_{20*})(-3+15e_{32}+8e_{42})g_{10*}$$
$$+(-10e_{12}-3e_{22}+15e_{32}+8e_{42})g_{20*}]\}+8\delta[-(e_{12}g_{20*})+2e_{32}(g_{10*}+g_{20*})]$$
$$+d^2\{(g_{10*}+g_{20*})[(5+34e_{32})g_{10*}+(-15e_{12}+5e_{22}+34e_{32})g_{20*}]$$
$$+16\delta[9e_{32}(g_{10*}+g_{20*})+2(-2e_{12}g_{20*}+e_{42}(g_{10*}+g_{20*})]\},$$

$$g_{12d}=g_{11d},$$

$$g_{22n}=-\{(-1+d^2)[-3(-4-d+4d^2+d^3)g_{10*}[2(-2+d^2)g_{10*}-(4-3d+d^2)g_{20*}]$$
$$\times[(2+6e_{32}+2d^2e_{32}-2e_{42}+d(-1+6e_{32}+3e_{42})]g_{10*}-[(2+3d+d^2)e_{12}$$
$$+(-2+d)e_{22}-6e_{32}-6de_{32}-2d^2e_{32}+2e_{42}-3de_{42}]g_{20*}\}$$
$$+d(4+d)\{[-4+6e_{32}+3d^2e_{32}+6e_{42}+d(-2+12e_{32}+3e_{42})]g_{10*}^2$$
$$-(10+4e_{12}+2d^2e_{12}+4e_{22}+6e_{32}-18e_{42}+d(-1+8e_{12}+2e_{22}-6e_{32}+3e_{42})\}g_{10*}g_{20*}$$
$$+[(2+d+d^2)e_{12}+(-10+d)e_{22}-3(4e_{32}+2de_{32}+d^2e_{32}-4e_{42}+2de_{42})]g_{20*}^2$$
$$\times[-8(2+d)\epsilon+3(-1+d)^2(1+d)(2g_{10*}+g_{20*})],$$

$$g_{22d}=g_{21d},$$

$$e_{11n}=(g_q-g_sp_2)[g_pg_s(m_4n_2-m_3n_3)p_1+g_{10*}g_o[(m_4n_1+m_1n_3)p_4-(m_3n_1+m_1n_2)p_5],$$

$$e_d=g_s^3(m_4n_2-m_3n_3)p_1p_2+g_{20*}g_og_q[-(m_4n_1+m_1n_3)p_4+(m_3n_1+m_1n_2)p_5]$$
$$+g_{20*}g_og_s[(m_1n_3p_2+m_4n_1(-p_1+p_2)]p_4+m_3n_1p_1p_5-m_3n_1p_2p_5-m_1n_2p_2p_5+m_2p_1(n_3p_4-n_2p_5))$$
$$+g_k\{g_qg_s[-(m_4n_2)+m_3n_3]+g_{20*}g_o(m_4n_1p_4-m_2n_3p_4-m_3n_1p_5+m_2n_2p_5)\},$$

$$e_{12n}=(g_q-g_sp_2)[g_pg_s(m_4n_2-m_3n_3)p_2+g_{10*}g_o(-m_4n_1p_4+m_2n_3p_4+m_3n_1p_5-m_2n_2p_5)],$$

$$e_{21n}=(g_k-g_sp_1)[g_pg_s(m_4n_2-m_3n_3)p_1+g_{10*}g_o(m_4n_1p_4+m_1n_3p_4-m_3n_1p_5-m_1n_2p_5)],$$

$$e_{22n}=(g_k-g_sp_1)(g_pg_s(m_4n_2-m_3n_3)p_2+g_{10*}g_o[-m_4n_1p_4+m_2n_3p_4+m_3n_1p_5-m_2n_2p_5)],$$

$$e_{31n}=g_{20*}g_pg_sp_1(-g_km_4n_1+g_qm_4n_1+g_qm_1n_3+g_km_2n_3+g_sm_4n_1p_1-g_sm_2n_3p_1-g_sm_4n_1p_2-g_sm_1n_3p_2)$$
$$+g_{10*}\{g_sp_1[g_s^2(m_4n_1+m_1n_3)p_2-g_{20*}g_o(m_1+m_2)n_1p_5]+g_k\{-[g_qg_s(m_4n_1+m_1n_3)]+g_{20*}g_o(m_1+m_2)n_1p_5)\},$$

$$e_{32n}=g_{20*}g_pg_sp_2(-g_km_4n_1+g_qm_4n_1+g_qm_1n_3+g_km_2n_3+g_sm_4n_1p_1-g_sm_2n_3p_1-g_sm_4n_1p_2-g_sm_1n_3p_2)$$
$$+g_{10*}[g_kg_qg_s(m_4n_1-m_2n_3)+g_s^3(-m_4n_1+m_2n_3)p_1p_2-g_{20*}g_og_q(m_1+m_2)n_1p_5+g_{20*}g_og_s(m_1+m_2)n_1p_2p_5],$$

$$e_{41n}=g_{20*}g_pg_sp_1(-g_km_3n_1+g_qm_3n_1+g_qm_1n_2+g_km_2n_2+g_sm_3n_1p_1-g_sm_2n_2p_1-g_sm_3n_1p_2-g_sm_1n_2p_2)$$
$$+g_{10*}(g_sp_1[g_s^2(m_3n_1+m_1n_2)p_2-g_{20*}g_o(m_1+m_2)n_1p_4]+g_k\{-[g_qg_s(m_3n_1+m_1n_2)]+g_{20*}g_o(m_1+m_2)n_1p_4\}),$$

$$e_{42n}=g_{20*}g_pg_sp_2(-g_km_3n_1+g_qm_3n_1+g_qm_1n_2+g_km_2n_2+g_sm_3n_1p_1-g_sm_2n_2p_1-g_sm_3n_1p_2-g_sm_1n_2p_2)$$
$$+g_{10*}[g_kg_qg_s(m_3n_1-m_2n_2)+g_s^3(-m_3n_1+m_2n_2)p_1p_2-g_{20*}g_og_q(m_1+m_2)n_1p_4+g_{20*}g_og_s(m_1+m_2)n_1p_2p_4],$$

where

$$l_1=24+16d-22d^2-16d^3-2d^4,$$

$$m_1=48-16d-52d^2+16d^3+4d^4,$$

$$m_2=-48-80d+60d^2+96d^3-10d^4-16d^5-2d^6,$$

$$m_3=-48+112d+32d^2-130d^3+14d^4+18d^5+2d^6,$$

$$m_4=48+104d-62d^2-127d^3+11d^4+23d^5+3d^6,$$

$$n_1=48+56d-40d^2-56d^3-8d^4,$$

$$n_2=48+104d-32d^2-104d^3-16d^4,$$

$$n_3=-48+16d+10d^2-41d^3+35d^4+25d^5+3d^6,$$

$$o_1=26-7d-27d^2+7d^3+d^4,$$

$$o_2=-12+12d^2,$$

$$p_1=-96-64d+88d^2+64d^3+8d^4,$$

$$p_2=-96-64d+124d^2+82d^3-26d^4-18d^5-2d^6,$$

$$p_3=96+40d-140d^2-60d^3+42d^4+20d^5+2d^6,$$

$$p_4=144+96d-132d^2-96d^3-12d^4,$$

$$p_5=144+96d-186d^2-123d^3+39d^4+27d^5+3d^6,$$

$$p_6=-24d-52d^2+4d^3+50d^4+20d^5+2d^6,$$

$$p_7=96+16d-192d^2-56d^3+92d^4+40d^5+4d^6,$$

$$q_1=96+16d-156d^2-38d^3+58d^4+22d^5+2d^6,$$

$$r_1=24-4d-36d^2+2d^3+12d^4+2d^5,$$

$$r_2=12+2d-18d^2-3d^3+6d^4+d^5,$$

$$r_3=12-6d-18d^2+5d^3+6d^4+d^5,$$

$$g_s=g_{10*}+g_{20*},$$

$$g_p=g_{10*}+g_{10*}^2/g_{20*},$$

$$g_k=(g_{10*}^2p_3)/g_{20*}+g_{20*}p_6+g_{10*}p_7,$$

$$g_q=(-(dg_{20*}^2l_1)+g_{10*}(g_{10*}p_3+g_{20*}q_1))/g_{20*},$$

$$g_o=g_s^2/g_{20*}.$$

The stability of the fixed point is determined by the 2×2 block of the stability matrix, which corresponds to the β functions of α_5 and χ_3. The eigenvalue which responds for instability has the form

$$\lambda=\lambda_0+\lambda_1\alpha_1+\lambda_2\alpha_2,$$

where

$$\lambda_0=\frac{dg_{20*}(g_{10*}+g_{20*})o_1-\sqrt{t_1}+g_{10*}g_{20*}r_1+g_{10*}^2r_2+g_{20*}^2r_3}{8d(12+8d^2+d^2)g_{20*}},$$

$$\lambda_1=\frac{\lambda_{1n}}{\lambda_d},$$

$$\lambda_{1n}=dg_{20*}^2(g_{11*}+g_{21*})o_1\sqrt{t_1}+g_{20*}[-t_2+g_{11*}\sqrt{t_1}(g_{20*}r_1+2g_{10*}r_2)]+g_{21*}\sqrt{t_1}(\sqrt{t_1}-g_{10*}^2r_2+g_{20*}^2r_3),$$

$$\lambda_d=8d(12+8d+d^2)g_{20*}^2\sqrt{t_1},$$

$$\lambda_2=\frac{\lambda_{2n}}{\lambda_d},$$

$$\lambda_{2n}=dg_{20*}^2(g_{12*}+g_{22*})o_1\sqrt{t_1}+g_{20*}[-t_3+g_{12*}\sqrt{t_1}(g_{20*}r_1+2g_{10*}r_2)]+g_{22*}\sqrt{t_1}(\sqrt{t_1}-g_{10*}^2r_2+g_{20*}^2r_3),$$

with

$$\begin{aligned}t_1={}&d^2g_{20*}^2(g_{10*}+g_{20*})^2(o_1^2-4o_2^2)\\&-2dg_{20*}o_1(g_{10*}+g_{20*})\\&\times(g_{10*}g_{20*}r_1+g_{10*}^2r_2+g_{20*}^2r_3)\\&+(g_{10*}g_{20*}r_1+g_{10*}^2r_2+g_{20*}^2r_3)^2,\end{aligned}$$

$$\begin{aligned}t_2={}&2(d^2g_{20*}(g_{10*}+g_{20*})(g_{11*}g_{20*}+(g_{10*}\\&+2g_{20*})g_{21*})(o_1^2-4o_2^2)+(g_{10*}g_{20*}r_1+g_{10*}^2r_2\\&+g_{20*}^2r_3)(g_{11*}g_{20*}r_1+g_{10*}g_{21*}r_1+2g_{10*}g_{11*}r_2\\&+2g_{20*}g_{21*}r_3)-do1\{g_{10*}^3g_{21*}r_2+g_{10*}^2g_{20*}[3g_{11*}r_2\\&+2g_{21*}(r_1+r_2)]+g_{20*}^3[4g_{21*}r_3+g_{11*}(r_1+r_3)]\\&+g_{10*}g_{20*}^2[2g_{11*}(r_1+r_2)+3g_{21*}(r_1+r_3)]\},\end{aligned}$$

$$\begin{aligned}t_3={}&2(d^2g_{20*}(g_{10*}+g_{20*})[g_{12*}g_{20*}+(g_{10*}+2g_{20*})g_{22*}]\\&\times(o_1^2-4o_2^2)+(g_{10*}g_{20*}r_1+g_{10*}^2r_2+g_{20*}^2r_3)\\&\times(g_{12*}g_{20*}r_1+g_{10*}g_{22*}r_1+2g_{10*}g_{12*}r_2\\&+2g_{20*}g_{22*}r_3)-do_1\{g_{10*}^3g_{22*}r_2+g_{10*}^2g_{20*}[3g_{12*}r_2\\&+2g_{22*}(r_1+r_2)]+g_{20*}^3[4g_{22*}r_3+g_{12*}(r_1+r_3)]\\&+g_{10*}g_{20*}^2[2g_{12*}(r_1+r_2)+3g_{22*}(r_1+r_3)]\}).\end{aligned}$$

The borderline dimension d_c is defined as a solution of the equation

$$\lambda(d_c,\epsilon,\alpha_1,\alpha_2)=0.$$

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